

Integral equation approach for the numerical solution of a Robin problem for the Klein-Gordon equation in a doubly connected domain

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Abstract

In this paper we consider a Robin problem for the Klein-Gordon equation in a doubly-connected domain. The solution domain considered is a bounded smooth doubly connected planar domain bounded by two simple disjoint closed curves. The analysis of the problem is based on the indirect integral equations method. The solution is represented as a sum of two single-layer potentials defined on each of the two boundary curves with unknown densities. To find out the densities the representation is matched with the given Robin data to generate a system of linear integral equations of the second kind with continuous and weakly-singular kernels. It is shown that the operator corresponding to this system is injective and due to its compactness according to Riesz theory there exists a unique solution. To discretize the system we apply Nyström method with a specifically chosen quadrature rules to obtain an exponential order of convergence of the approximate solution. Numerical experiments are conducted for three testing examples that back up the theoretical reasoning.

1. Introduction

Integral equation method became widely used about a century ago in the treatment of boundary value problems and is a classic approach in mathematical physics. It allows us to transform a boundary value problem to an integral equation and has a range of indisputable advantages: reducing the dimension of the problem, applicability to domains with a complicated shape. It is applied not only to solve boundary problems but also to provide an adequate theoretical background. Reducing the boundary problem to an integral equation we can operate within

Riesz theory to establish the existence and uniqueness of a solution.

Once a boundary integral equation is defined on a smooth curve, there are many numerical methods to solve it. The numerical analysis of most such problems is now well-understood and, as usually, easier to apply and cost less computational effort than those straightforwardly applied to the initial problem. It is based on quadrature approximation of integrals and theory of compact operators. Thus, we will seek the best way to approximate integrals in a resulting mathematical model and show that it directly affects the rate of convergence of the approximate solution.

A great deal of this research has been conducted on the use of boundary integral equations method for Laplace equation. Though, to introduce some variety in this paper we consider the time independent Klein-Gordon equation which occurs in relativistic quantum field theory. The problem and solution domain are as described below.

Let $D_1 \subset \mathbb{R}^2$ be a simply connected sufficiently smooth bounded domain with boundary $\Gamma_1 \in C^2$, and let $D_2 \subset D_1$ be a simply connected (smooth) bounded domain with boundary $\Gamma_2 \in C^2$. We define $D = D_1 \setminus \overline{D_2}$.

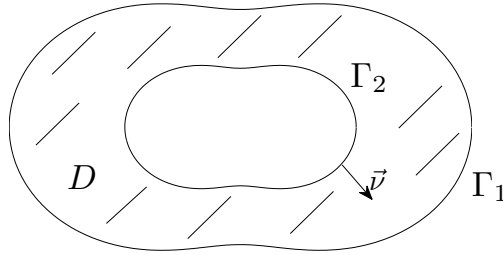


Figure 1: ' D

We consider the following Robin problem of finding a function $u : D \rightarrow \mathbb{R}$, $u \in C^2(D) \cap C^1(\overline{D})$, such that

$$\Delta u - \kappa^2 u = 0 \quad \text{in } D \quad (1.1)$$

with the boundary conditions

$$\frac{\partial u}{\partial \nu} + \lambda_i u = f_i \quad \text{on } \Gamma_i, \quad (1.2)$$

where $f_i, \lambda_i > 0$, $i = 1, 2$ – known functions, $\kappa > 0$ – known constant and ν – the outward unit normal to the boundary of D .

To construct the solution to (1.1)(1.2), we shall represent the solution as a sum of two single-layer potentials over Γ_1 and Γ_2 , respectively, with both the densities in this representation being unknown. To find these densities we use the representation of the solution to match up the given Robin data. This approach results in a system of boundary integral equations to be solved for the unknown densities. Moreover, the operator corresponding to our system is injective, therefore, according to Riesz theory, bijective and the system has a unique solution. For the discretisation of the system of integral equations, we employ the Nyström method,

splitting the kernels appropriately to take into account the singularities, which generate a linear system to solve. Once the (discrete) densities have been obtained, they can be used to find an approximation to the solution of (1.1)(1.2) and in particular the value of the solution on the boundaries of the domain D .

For the outline of this work, in Section 2 we present the indirect integral method for the Robin problem (1.1)-(1.2) leading to a system to solve for two densities. We also show how to discretise the obtained system and discuss the solution of it via the Nyström method. In Section 3, numerical examples are presented. The results obtained show that accurate approximations can be obtained with the proposed indirect approach and with little computational effort.

2. An indirect integral equation method for the Robin problem (1.1) (1.2)

In this section, we present an indirect integral equation method for the Robin problem (1.1)(1.2). It is known from [2], that the fundamental solution to Klein-Gordon equation $\Delta u - \kappa^2 u = 0$ \mathbb{R}^2 has the form:

$$\Phi(x, y) = \frac{1}{2\pi} K_0(\kappa |x - y|), \quad x \neq y,$$

where K_0 is a modified Bessel function[2]. We represent the solution u of (1.1)(1.2) in the form of a sum of single-layer potentials

$$u(x) = \int_{\Gamma_1} \varphi_1(y) \Phi(x, y) ds(y) + \int_{\Gamma_2} \varphi_2(y) \Phi(x, y) ds(y), \quad x \in D \quad (2.1)$$

with unknown densities $\varphi_1 \in C(\Gamma_1)$ and $\varphi_2 \in C(\Gamma_2)$. Clearly the function u in (2.1) satisfies the Klein-Gordon equation (1.1). Matching this representation with the given Robin data in (1.2) and using well-known properties for the restriction of these single-layer potentials and their derivatives on the boundary of the domain D , we get the following system of integral equations

$$\left\{ \begin{array}{l} \frac{1}{2} \varphi_1(x) + \int_{\Gamma_1} \varphi_1(y) \left(\frac{\partial \Phi(x, y)}{\partial \nu(x)} + \lambda_1 \Phi(x, y) \right) ds(y) + \\ \int_{\Gamma_2} \varphi_2(y) \left(\frac{\partial \Phi(x, y)}{\partial \nu(x)} + \lambda_1 \Phi(x, y) \right) ds(y) = f_1(x), \quad x \in \Gamma_1, \\ -\frac{1}{2} \varphi_2(x) + \int_{\Gamma_1} \varphi_1(y) \left(\frac{\partial \Phi(x, y)}{\partial \nu(x)} + \lambda_2 \Phi(x, y) \right) ds(y) + \\ \int_{\Gamma_2} \varphi_2(y) \left(\frac{\partial \Phi(x, y)}{\partial \nu(x)} + \lambda_2 \Phi(x, y) \right) ds(y) = f_2(x), \quad x \in \Gamma_2. \end{array} \right. \quad (2.2)$$

to be solved for the unknown densities φ_1 and φ_2 . We define the following operators:

$$\begin{aligned}
(S\varphi)(x) &:= 2 \int_{\Gamma_1} \varphi_1(y) \left(\frac{\partial \Phi(x, y)}{\partial \nu(x)} + \lambda_1 \Phi(x, y) \right) ds(y), \quad x \in \Gamma_1, \\
(N\varphi)(x) &:= 2 \int_{\Gamma_2} \varphi_2(y) \left(\frac{\partial \Phi(x, y)}{\partial \nu(x)} + \lambda_1 \Phi(x, y) \right) ds(y), \quad x \in \Gamma_1, \\
(H\varphi)(x) &:= -2 \int_{\Gamma_1} \varphi_1(y) \left(\frac{\partial \Phi(x, y)}{\partial \nu(x)} + \lambda_2 \Phi(x, y) \right) ds(y), \quad x \in \Gamma_2, \\
(C\varphi)(x) &:= -2 \int_{\Gamma_2} \varphi_2(y) \left(\frac{\partial \Phi(x, y)}{\partial \nu(x)} + \lambda_2 \Phi(x, y) \right) ds(y), \quad x \in \Gamma_2.
\end{aligned} \tag{2.3}$$

then the system (2.1) can be written as

$$\varphi - U\varphi = F, \tag{2.4}$$

where $\varphi := (\varphi_1, \varphi_2)^T$, $F := (2f_1, -2f_2)^T$ and U is defined as

$$U := \begin{pmatrix} S & N \\ H & C \end{pmatrix}.$$

Let us denote $\Gamma := \Gamma_1 \cup \Gamma_2$. The operators N, H have continuous kernels and S, C have weakly singular kernels, thus they are compact on $C(\Gamma)$.

It is easily seen that $N(I - U)$ nullspace contains only the trivial element, thus $(I - U)$ is injective and, according to Riesz theory, $(I - U)$ bijective, that means

$$\forall F \in C(\Gamma) \quad \exists! \varphi \in C(\Gamma) : \quad (I - U)\varphi = F.$$

Furthermore the inverse operator $(I - U)^{-1}$ is bounded and the solution of the equation (2.4) gains the following form

$$\varphi = (I - U)^{-1}F. \quad \square$$

2.1 Parametrisation of the system of integral equations

Let the boundaries Γ_1 and Γ_2 have the parametric representation

$$\Gamma_i := \{x_i(t) = (x_{i1}(t), x_{i2}(t)), \quad t \in [0, 2\pi]\}, \tag{2.5}$$

where $x_i: \mathbb{R} \rightarrow \mathbb{R}^2$, are 2π - periodic with $|x'_i(t)| > 0$ for all $t \in [0, 2\pi]$, $x_i \in C^2([0, 2\pi] \times [0, 2\pi])$, $i = 1, 2$.

Considering that

$$\frac{\partial K_0(\kappa|x-y|)}{\partial \nu(x)} = -\kappa K_1(\kappa|x-y|) \frac{(x-y) \cdot \nu(x)}{|x-y|},$$

(where K_1 is another modified Bessel function from [2]) and using parametric representations

(2.5) in , we obtain the parametrised system of integral equations

$$\begin{cases} \frac{1}{2} \frac{\psi_1(t)}{|x'_1(t)|} + \frac{1}{2\pi} \int_0^{2\pi} \psi_1(\tau) H_{11}(t, \tau) d\tau + \frac{1}{2\pi} \int_0^{2\pi} \psi_2(\tau) H_{12}(t, \tau) d\tau = g_1(t), \\ -\frac{1}{2} \frac{\psi_2(t)}{|x'_2(t)|} + \frac{1}{2\pi} \int_0^{2\pi} \psi_1(\tau) H_{21}(t, \tau) d\tau + \frac{1}{2\pi} \int_0^{2\pi} \psi_2(\tau) H_{22}(t, \tau) d\tau = g_2(t), \end{cases} \quad (2.6)$$

where $\psi_i(t) = \varphi_i(x_i(t)) |x'_i(t)|$, $g_i(t) = f_i(x_i(t))$, $i = 1, 2$ and the kernels are given by

$$H_{ij}(t, \tau) = -\kappa K_1(\kappa|x-y|) \frac{(x-y) \cdot \nu(x)}{|x-y|} + \lambda_i K_0(\kappa|x-y|) \Big|_{\substack{x=x_i(t) \\ y=x_j(\tau)}}, \quad t \neq \tau, \quad i, j = 1, 2.$$

The kernels $H_{ii}(t, \tau)$, $i = 1, 2$ have logarithmic singularities when $\tau \rightarrow t$. We will decompose the kernels to exclude the singularity

$$H_{ii}(t, \tau) = H_{ii}^{(1)}(t, \tau) \ln \frac{4}{e} \sin^2 \frac{t-\tau}{2} + H_{ii}^{(2)}(t, \tau),$$

with $H_{ii}^{(1)}, H_{ii}^{(2)}$ – smooth functions of the form

$$\begin{aligned} H_{ii}^{(1)} &= -\frac{\lambda_i}{2} I_0(\kappa|x-y|) \Big|_{\substack{x=x_i(t) \\ y=x_i(\tau)}}, \quad t \neq \tau; \\ H_{ii}^{(2)}(t, \tau) &= H_{ii}(t, \tau) - H_{ii}^{(1)}(t, \tau) \ln \frac{4}{e} \sin^2 \frac{t-\tau}{2} = \\ &= \frac{\lambda_1}{2} I_0(\kappa|x-y|) \ln \frac{16 \sin^2 \frac{t-\tau}{2}}{e \kappa^2 |x-y|^2} - \\ &\quad - \frac{(x-y) \cdot \nu(x)}{|x-y|^2} - \kappa \ln \frac{\kappa|x-y|}{2} I_1(\kappa|x-y|) \frac{(x-y) \cdot \nu(x)}{|x-y|} - \\ &\quad - \kappa \Sigma_1(\kappa|x-y|) \frac{(x-y) \cdot \nu(x)}{|x-y|} + \\ &\quad + \lambda_1 \Sigma_0(\kappa|x-y|), \quad x = x_i(t), \quad y = x_i(\tau), \quad t \neq \tau; \end{aligned}$$

and with the diagonal terms

$$\begin{aligned} \lim_{\tau \rightarrow t} H_{ii}^{(1)}(t, \tau) &= -\frac{\lambda_i}{2}, \\ \lim_{\tau \rightarrow t} H_{ii}^{(2)}(t, \tau) &= \frac{x'_{i2}(t)x''_{i1}(t) - x'_{i1}(t)x''_{i2}(t)}{2|x'_i(t)|^3} + \frac{\lambda_i}{2} \ln \frac{4}{e \kappa^2 |x'_i(t)|^2} - \lambda_i \gamma, \end{aligned}$$

The solution to (1.1)-(1.2) then will gain a parametrized form in the domain D

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} \psi_1(\tau) \tilde{H}_1(x, \tau) d\tau + \frac{1}{2\pi} \int_0^{2\pi} \psi_2(\tau) \tilde{H}_2(x, \tau) d\tau, \quad x \in D, \quad (2.7)$$

where kernels \tilde{H}_1 i \tilde{H}_2 are given by:

$$\begin{aligned} \tilde{H}_1(x, \tau) &= K_0(\kappa |x - x_1(\tau)|), \\ \tilde{H}_2(x, \tau) &= K_0(\kappa |x - x_2(\tau)|). \end{aligned}$$

On the boundary Γ_i , $i = 1, 2$ we have:

$$u(x_i(t)) = \frac{1}{2\pi} \int_0^{2\pi} \psi_1(\tau) \hat{H}_1(t, \tau) d\tau + \frac{1}{2\pi} \int_0^{2\pi} \psi_2(\tau) \hat{H}_2(t, \tau) d\tau, \quad (2.8)$$

where kernels are given by

$$\hat{H}_j(t, \tau) = K_0(\kappa |x_i(t) - x_j(\tau)|), \quad t \neq \tau, \quad j = 1, 2.$$

When $i = j$, a singularity appears in \hat{H}_i , thus we will exclude it similary as in (2.7)

$$\hat{H}_i(t, \tau) = \hat{H}_i^{(1)}(t, \tau) \ln \left(\frac{4}{e} \sin^2 \frac{t - \tau}{2} \right) + \hat{H}_i^{(2)}(t, \tau),$$

where

$$\hat{H}_i^{(2)}(t, \tau) = \begin{cases} K_0(\kappa |x_i(t) - x_i(\tau)|) - \hat{H}_i^{(1)}(t, \tau) \ln \left(\frac{4}{e} \sin^2 \frac{t - \tau}{2} \right), & t \neq \tau; \\ \frac{\lambda_i}{2} \ln \frac{4}{e\kappa^2 |x'_i(t)|^2} - \lambda_i \gamma, & t = \tau; \end{cases} \quad (2.9)$$

$$\hat{H}_i^{(1)}(t, \tau) = -\frac{\lambda_i}{2} I_0(\kappa |x_i(t) - x_i(\tau)|).$$

2.2 Discretisation of the parametrised system (2.6)

We consider two quadrature rules constructed via trigonometrical interpolation with $2M$ equidistant nodal points $t_j = \frac{j\pi}{M}$, $j = 0 \dots 2M - 1$, $M \in \mathbb{N}$

$$\frac{1}{2\pi} \int_0^{2\pi} f(\tau) d\tau \approx \frac{1}{2M} \sum_{j=0}^{2M-1} f(t_j), \quad (2.10)$$

$$\frac{1}{2\pi} \int_0^{2\pi} f(\tau) \ln \left(\frac{4}{e} \sin^2 \frac{t - \tau}{2} \right) d\tau \approx \sum_{j=0}^{2M-1} R_j(t) f(t_j). \quad (2.11)$$

with corresponding weight functions [1]

$$R_j(t) = -\frac{1}{M} \sum_{m=1}^{M-1} \frac{1}{m} \cos m(t - t_j) - \frac{1}{2M^2} \cos M(t - t_j).$$

Using the Nyström method with quadratures (2.10) and (2.11) to the integral equations (2.6), we obtain the following system of linear equations

$$\begin{cases} \frac{1}{2} \frac{\tilde{\psi}_{1,i}}{|x'_1(t_i)|} + \sum_{j=0}^{2M-1} \tilde{\psi}_{1,j} \left[R_j(t_i) H_{11}^{(1)}(t_i, t_j) + \frac{1}{2M} H_{11}^{(2)}(t_i, t_j) \right] + \frac{1}{2M} \sum_{j=0}^{2M-1} \tilde{\psi}_{2,j} H_{12}(t_i, t_j) = g_1(t_i), \\ -\frac{1}{2} \frac{\tilde{\psi}_{2,i}}{|x'_2(t_i)|} + \frac{1}{2M} \sum_{j=0}^{2M-1} \tilde{\psi}_{1,j} H_{21}(t_i, t_j) + \frac{1}{2M} \sum_{j=0}^{2M-1} \tilde{\psi}_{2,j} \left[R_j(t_i) H_{22}^{(1)}(t_i, t_j) + \frac{1}{2M} H_{22}^{(2)}(t_i, t_j) \right] = g_2(t_i) \end{cases} \quad (2.12)$$

to be solved for $\tilde{\psi}_{k,j} \approx \psi_k(t_j)$, $k = 1, 2$, $i = 0 \dots 2M - 1$.

After applying the quadrature rules (2.10), (2.11) to approximate the integrals (2.7) and (2.8) the approximate solution to (1.1)-(1.2) in the domain D can be found as

$$u(x) \approx \frac{1}{2M} \sum_{j=0}^{2M-1} \tilde{\psi}_{1,j} \tilde{H}_1(x, t_j) + \frac{1}{2M} \sum_{j=0}^{2M-1} \tilde{\psi}_{2,j} \tilde{H}_2(x, t_j), \quad x \in D,$$

On the boundary Γ_1

$$u(x_1(t)) \approx \frac{1}{2M} \sum_{j=0}^{2M-1} \tilde{\psi}_{1,j} \left\{ R_j(t) \hat{H}_1^{(1)}(t, t_j) + \frac{1}{2M} \hat{H}_1^{(2)}(t, t_j) \right\} + \sum_{j=0}^{2M-1} \tilde{\psi}_{2,j} \tilde{H}_2(x_1(t), t_j),$$

and Γ_2 correspondingly

$$u(x_2(t)) \approx \frac{1}{2M} \sum_{j=0}^{2M-1} \tilde{\psi}_{1,j} \tilde{H}_1(x_2(t), t_j) + \sum_{j=0}^{2M-1} \tilde{\psi}_{2,j} \left\{ R_j(t) \hat{H}_2^{(1)}(t, t_j) + \frac{1}{2M} \hat{H}_2^{(2)}(t, t_j) \right\}.$$

2.3 Convergence and error analysis

Convergence of the method proposed above is based on the theory of compact operators and error estimate of trigonometric interpolation in Banach spaces. It is known [1], that error

$$R_T(g) = \frac{1}{2\pi} \int_0^{2\pi} g(t) dt - \frac{1}{2n} \sum_{j=0}^{2n-1} g\left(\frac{j\pi}{n}\right)$$

of the composed trapezoidal rule for analytical 2π - periodic function g may be evaluated as following

$$|R_T(g)| \leq C e^{-2n\sigma}, \quad (2.13)$$

where C i σ - positive constants, depending on g .

We approximate the following operators S, N, H, C , presented in (2.3), with sequences of approximating operators:

$$\begin{aligned}
(S_n\varphi)(x) &:= 2 \sum_{k=1}^{2n-1} K_S(x, x_k^{(n)}) \varphi(x_k^{(n)}), \quad x \in \Gamma_1, \\
(N_n\varphi)(x) &:= 2 \sum_{k=1}^{2n-1} K_N(x, x_k^{(n)}) \varphi(x_k^{(n)}), \quad x \in \Gamma_1, \\
(H_n\varphi)(x) &:= -2 \sum_{k=1}^{2n-1} K_H(x, x_k^{(n)}) \varphi(x_k^{(n)}), \quad x \in \Gamma_2, \\
(C_n\varphi)(x) &:= -2 \sum_{k=1}^{2n-1} K_C(x, x_k^{(n)}) \varphi(x_k^{(n)}), \quad x \in \Gamma_2,
\end{aligned} \tag{2.14}$$

let us define an operator U_n

$$U_n := \begin{pmatrix} S_n & N_n \\ H_n & C_n \end{pmatrix}. \tag{2.15}$$

Then the solution to the following equation

$$\varphi - U\varphi = F, \tag{2.16}$$

is approximated through the solution to the approximate equation

$$\varphi_n - U_n\varphi_n = F, \tag{2.17}$$

which is reduced to a system of linear equations. As stated in [1], the rate of convergence of the approximate solution (2.17) to the exact solution (2.16) is ruled by the convergence of the sequences of approximating operators U_n to the operator U .

$$\|\varphi_n - \varphi\|_\infty \leq M \|U_n\varphi - U\varphi\|_\infty, \quad M > 0. \tag{*}$$

The error of trigonometric interpolation (2.13) gives for $\Gamma \in C^\infty$ $F \in C^\infty(\Gamma)$:

$$\|\varphi_n - \varphi\|_\infty = O(e^{-n\sigma}), \quad \sigma > 0.$$

3. Numerical examples

3.1 Example 1

As a test example we take a fundamental solution to the Klein-Gordon equation

$$f_1(x) = \frac{\partial \Phi(x, y^*)}{\partial \nu(x)} + \Phi(x, y^*) \quad \text{on } \Gamma_1, \quad f_2(x) = \frac{\partial \Phi(x, y^*)}{\partial \nu(x)} + \Phi(x, y^*) \quad \text{on } \Gamma_2, \tag{3.1}$$

where y^* - a point outside of the domain D , we take $\lambda_1 = \lambda_2 = \kappa = 1$. In this case the exact solution is

$$u(x) = \Phi(x, y^*), \quad x \in D.$$

3.1.1 Example 1.a

The doubly connected solution domain D is given in Fig.2 and the two boundary curves are described by the parametrization

$$\begin{aligned}\Gamma_1 &= \{x_1(t) = (1.3 \cos t, \sin t), t \in [0, 2\pi]\}; \\ \Gamma_2 &= \{x_2(t) = (0.5 \cos t, 0.4 \sin t - 0.3 \sin^2 t), t \in [0, 2\pi]\}.\end{aligned}\tag{3.2}$$

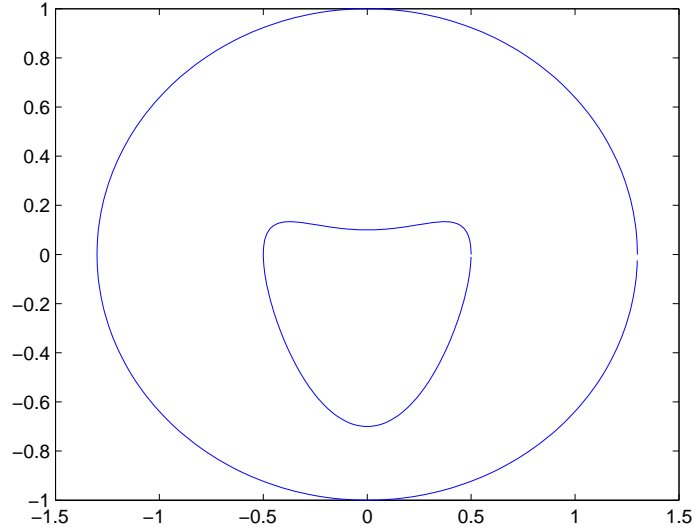


Figure 2: Boundary curves Γ_1 and Γ_2 in case (3.2)

Functions f_1 and f_2 are given as in (3.4), where $y^* = (4, 0)$. Absolute error of the exact and approximate solution to (1.1)–(1.2) problem at some points $x \in D$ depending on the amount of quadrature nodes are given in Tab.1

Tab.1

Absolute error at some points $x \in D$ in case (3.2)

M	$x = (0.0, 0.5)$	$x = (1, 0)$	$x = (-0.5, 0.4)$	$x = (-0.5, -0.2)$
4	0.00002033096073197	0.00026101137354300	0.00001000429363413	0.00002914772071645
8	0.00000009457990411	0.00000436317651679	0.00000013044431753	0.00000042154352825
16	0.00000000001990384	0.00000000388276707	0.00000000004216076	0.00000000017064445
32	0.00000000000000073	0.00000000000000870	0.00000000000000041	0.00000000000000037
64	0.00000000000000009	0.00000000000000019	0.00000000000000008	0.00000000000000009

3.1.2 Example 1.b

Let the boundary curves Γ_1 and Γ_2 have the following parametrization Fig.3:

$$\begin{aligned}\Gamma_1 &= \{x_1(t) = (2 \cos t, 2 \sin t), t \in [0, 2\pi]\}; \\ \Gamma_2 &= \{x_2(t) = (0.5 \cos t, 0.5 \sin t), t \in [0, 2\pi]\}.\end{aligned}\tag{3.3}$$

Functions f_1 and f_2 are given as in (3.4), where $y^* = (4, 0)$. Absolute errors are presented in Tab. 2, graphs of the approximate solution are shown on Fig.4

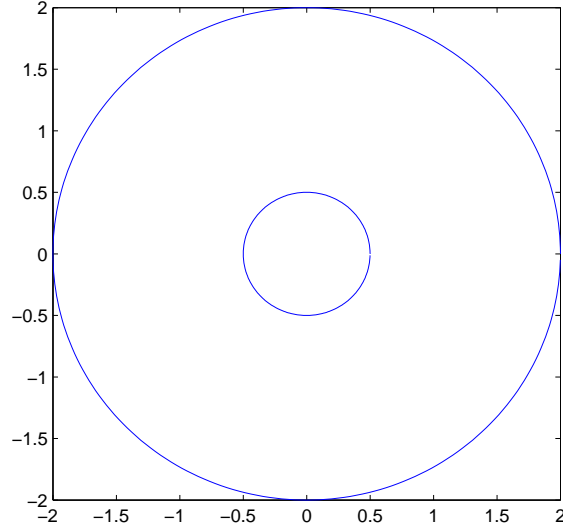


Figure 3: Boundary curves Γ_1 and Γ_2 in case (3.3)

Tab. 2

Absolute errors for some $x \in D$ in case (3.3)

M	$x = (1, 1)$	$x = (-1, 0.7)$	$x = (0, -1, 5)$	$x = (1.8, -0.3)$
4	0.00087841425733911	0.00070385813517727	0.00403335542963912	0.00644178058184269
8	0.00006457661678542	0.00001800029342743	0.00403335542963912	0.00018686761910423
16	0.00000008468542928	0.00000002642802893	0.00000005213821305	0.00000027397602064
32	0.00000000000011986	0.0000000000004097	0.00000000000010210	0.00000000000055763
64	0.00000000000000192	0.0000000000000014	0.0000000000000012	0.00000000000000186

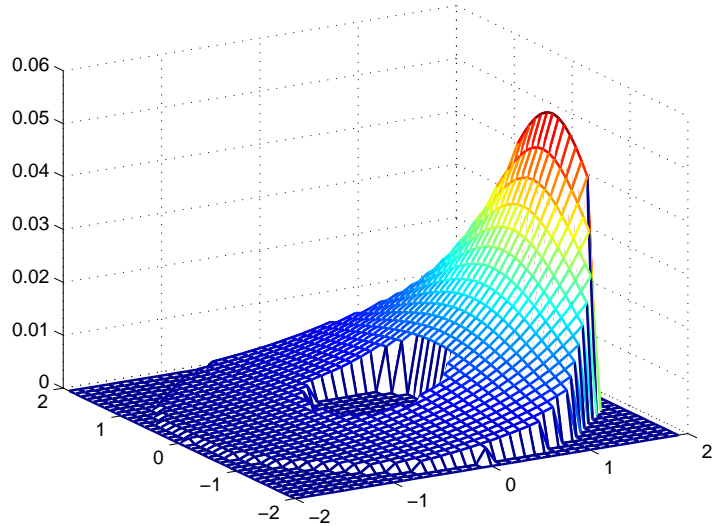


Figure 4: Graph of the approximate solution in case (3.3)

3.2 Example 2

Let the boundary curves Γ_1 and Γ_2 be the same as in Example 1.a, and the functions f_1 and f_2 are

$$f_1(x) = x_1^2 + x_2 \quad \text{on } \Gamma_1, \quad f_2(x) = x_1 + x_2^2 \quad \text{on } \Gamma_2, \quad (3.4)$$

Values of the approximate solution to (1.1)–(1.2) problem at some points $x \in D$ depending on the amount of quadrature nodes are given in Tab. 3

Tab. 3

Approximate solution values at some points $x \in D$

M	$x = (0, 0.4)$	$x = (1, 0)$	$x = (-0.5, 0.4)$	$x = (-0.6, -0.4)$
4	1.064989834	0.690396742	1.088551277	0.515033033
8	1.087775529	0.709894802	1.086481731	0.606838275
16	1.088553459	0.710216416	1.087487257	0.609990277
32	1.088551632	0.710212588	1.087487087	0.609982308
64	1.088551277	0.710212073	1.087486912	0.609981936

Discussion. From the results in tables in Example 1 it can be easily seen that orders of convergence are as predicted in the section 2.3. Though, practically, such rapid convergence would not have place for all input data. We shell point out that taking the point y^* in (3.4) close to the solution domain will increase the error between the approximate and exact solution and worsen the rate of convergence since bessel function K_0 becomes more "close" to singular in this case.

Conclusions

We have proposed an indirect integral equation method for the solution of the Robin problem for the Klein-Gordon equation in a doubly connected planar domain. The solution is represented as a sum of two single-layer potentials with unknown densities. To construct these densities the solution is matched up with the given Robin data. The Nyström method was employed for the discretisation of the integrals.

The numerical results agreed with a priori error estimates, thus supporting the theoretical reasoning. The obtained result points out the advantages of the indirect integral equation method, such as reducing the dimension of a problem and ability to be applied to a wider range of problems due to its applicability to problems with various domains. Furthermore, we have shown that one would benefit from implementing a trapezoidal quadrature rule in the Nyström method for the mentioned problem, since an approximate solution has then an exponential convergence rate.

The proposed approach can be applied also to exterior problems and domains in \mathbb{R}^3 . This is however deferred to a future work.

References

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